

NON GROUP-THEORETICAL SEMISIMPLE HOPF ALGEBRAS FROM GROUP ACTIONS ON FUSION CATEGORIES

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ABSTRACT. Given an action of a finite group G on a fusion category \mathcal{C} we give a criterion for the category of G -equivariant objects in \mathcal{C} to be group-theoretical, i.e., to be categorically Morita equivalent to a category of group-graded vector spaces. We use this criterion to answer affirmatively the question about existence of non group-theoretical semisimple Hopf algebras asked by P. Etingof, V. Ostrik, and the author in [ENO]. Namely, we show that certain $\mathbb{Z}/2\mathbb{Z}$ -equivariantizations of fusion categories constructed by D. Tambara and S. Yamagami [TY] are equivalent to representation categories of non group-theoretical semisimple Hopf algebras. We describe these Hopf algebras as extensions and show that they are upper and lower semisolvable.

1. INTRODUCTION

1.1. Conventions. Throughout this article we work over an algebraically closed field k of zero characteristic. All cocycles appearing in this article have coefficients in the trivial module k^\times . Module categories are assumed to be k -linear and semisimple with finite-dimensional Hom-spaces and finitely many isomorphism classes of simple objects. Functors between fusion categories and their module categories are assumed to be additive and k -linear. By a subcategory we always mean a full subcategory.

1.2. Main result. A tensor category is said to be *group-theoretical* [O2, ENO] if it is dual to a category of group-graded vector spaces with respect to an indecomposable module category, see Section 2.1 for definitions. Group-theoretical categories can be explicitly described in terms of finite groups and their cohomology, see [O2] and Section 2.2 below. Such categories were extensively studied, see e.g., [DGNO, EGO, ENO, GN, Na, NaNi, Nt], and many classification results were obtained. In particular, representation categories of semisimple quasi-Hopf algebras of dimension p^n , $n = 1, 2, \dots$ and pq , where p and q are prime numbers, are group-theoretical by [DGNO] and [EGO], respectively. On the other hand, there exist semisimple quasi-Hopf algebras of dimension $2p^2$, where p is an odd prime, with non group-theoretical representation categories [ENO, Remark 8.48].

The main result of this article is existence of semisimple Hopf algebras with non group-theoretical representation categories. This answers a question [ENO, Question 8.45] of P. Etingof, V. Ostrik, and the author.

To establish this we construct a series of fusion categories of Frobenius-Perron dimension $4p^2$, where p is an odd prime, that are non group-theoretical and admit tensor functors to the category of vector spaces. By Tannakian formalism [U] these

categories are equivalent to representation categories of some semisimple Hopf algebras. The construction of these categories involves an *equivariantization* procedure [AG, G]: given an action of a finite group G on a tensor category \mathcal{C} one forms a tensor category \mathcal{C}^G of G -equivariant objects in \mathcal{C} . See Section 2.3 for definitions. Specifically, we take \mathcal{C} to be a group-theoretical Tambara-Yamagami fusion category [TY] of Frobenius-Perron dimension $2p^2$ and explicitly define an action of $G = \mathbb{Z}/2\mathbb{Z}$ on it. The corresponding category of equivariant objects is equivalent to the representation category of a non group-theoretical semisimple Hopf algebra of dimension $4p^2$.

1.3. Organization of the paper. Section 2 contains basic definitions related to fusion categories and their module categories, a description of group-theoretical categories, the equivariantization construction, and Tambara-Yamagami fusion categories. In Section 3, given an action of a group G on a fusion category \mathcal{C} , we define a *crossed product* category $\mathcal{C} \rtimes G$ and show that it is dual to the category \mathcal{C}^G of G -equivariant objects in \mathcal{C} . We prove in Theorem 3.5 that $\mathcal{C} \rtimes G$ and \mathcal{C}^G are group-theoretical if and only if there exists a G -invariant indecomposable \mathcal{C} -module category with a pointed dual. This criterion is used in Section 4, where we explicitly construct an action of $\mathbb{Z}/2\mathbb{Z}$ on a group-theoretical Tambara-Yamagami category \mathcal{C}_p of Frobenius-Perron dimension $2p^2$, where p is an odd prime, and show that there are no \mathcal{C}_p -module categories with pointed dual invariant with respect to this action. The equivariantization categories $\mathcal{C}_p^{\mathbb{Z}/2\mathbb{Z}}$ are non group-theoretical but admit tensor functors to the category of vector spaces. Hence, these categories are equivalent to representation categories of semisimple Hopf algebras. This shows that there is a series of semisimple Hopf algebras H_p of dimension $4p^2$, where p is an odd prime, with non group-theoretical representation categories. The algebraic structure of these Hopf algebras is analyzed in Section 5. We describe H_p as an extension of already known Hopf algebras and show that it is upper and lower semisolvable in the sense of [MW].

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2. PRELIMINARIES

2.1. Fusion categories and their module categories. A *fusion category* over k is a k -linear semisimple rigid tensor category [BK, K] with finitely many isomorphism classes of simple objects, finite-dimensional Hom-spaces, and simple unit object $\mathbf{1}$.

Let \mathcal{C} be a fusion category. For any object X in \mathcal{C} its *Frobenius-Perron dimension* $\text{FPdim}(X)$ is defined as the largest non-negative real eigenvalue of the matrix of multiplication by X in the Grothendieck semi-ring of \mathcal{C} , see [FK] and [ENO, Section 8]. The Frobenius-Perron dimension of \mathcal{C} is, by definition, the sum of squares of Frobenius-Perron dimensions of simple objects of \mathcal{C} and is denoted $\text{FPdim}(\mathcal{C})$. In the special case when $\mathcal{C} = \text{Rep}(H)$ is the representation category of a semisimple quasi-Hopf algebra H the Frobenius-Perron dimensions coincide with vector space dimensions, i.e., $\text{FPdim}(V) = \dim_k(V)$ for any finite-dimensional H -module V and $\text{FPdim}(\text{Rep}(H)) = \dim_k(H)$.

A fusion category \mathcal{C} is *graded* by a finite group G if there is a decomposition

$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$$

of \mathcal{C} into a direct sum of full Abelian subcategories such that \otimes maps $\mathcal{C}_g \times \mathcal{C}_h$ to \mathcal{C}_{gh} for all $g, h \in G$. Note that \mathcal{C}_e , where e is the identity element of G , is the fusion subcategory of \mathcal{C} . In this paper we consider only faithful gradings, i.e., such that $\mathcal{C}_g \neq 0$ for all $g \in G$. In this case one has $\text{FPdim}(\mathcal{C}) = |G| \text{FPdim}(\mathcal{C}_e)$, see [ENO].

A right \mathcal{C} -module category is a category \mathcal{M} together with an exact bifunctor $\otimes : \mathcal{M} \times \mathcal{C} \rightarrow \mathcal{M}$ and a natural family of isomorphisms $M \otimes (X \otimes Y) \cong (M \otimes X) \otimes Y$ and $M \otimes \mathbf{1} \cong M$, $X, Y \in \mathcal{C}$, $M \in \mathcal{M}$ satisfying certain coherence conditions. See [O1] for details and for definitions of \mathcal{C} -module functors and their natural transformations. The category of \mathcal{C} -module functors between right \mathcal{C} -module categories \mathcal{M}_1 and \mathcal{M}_2 will be denoted $\text{Func}(\mathcal{M}_1, \mathcal{M}_2)$.

For two \mathcal{C} -module categories \mathcal{M}_1 and \mathcal{M}_2 their *direct sum* $\mathcal{M}_1 \oplus \mathcal{M}_2$ has an obvious \mathcal{C} -module category structure. A module category is *indecomposable* if it is not equivalent to a direct sum of two non-trivial module categories. It was shown in [O1] that \mathcal{C} -module categories are completely reducible, i.e., given a \mathcal{C} -module subcategory \mathcal{N} of a \mathcal{C} -module category \mathcal{M} there is a unique \mathcal{C} -module subcategory \mathcal{N}' of \mathcal{M} such that $\mathcal{M} = \mathcal{N} \oplus \mathcal{N}'$. Consequently, any \mathcal{C} -module category \mathcal{M} has a unique, up to a permutation of summands, decomposition $\mathcal{M} = \bigoplus_{x \in S} \mathcal{M}_x$ into a direct sum of indecomposable \mathcal{C} -module categories.

A theorem due to V. Ostrik [O1] states that any right \mathcal{C} -module category is equivalent to the category of left modules over some algebra in \mathcal{C} . Namely, given a non-zero object V of an indecomposable \mathcal{C} -module category \mathcal{M} , the internal Hom $\underline{\text{Hom}}(V, V)$ defined by a natural isomorphism

$$(1) \quad \text{Hom}_{\mathcal{M}}(X \otimes V, V) \cong \text{Hom}_{\mathcal{C}}(X, \underline{\text{Hom}}(V, V)), \quad X \in \mathcal{C},$$

is an algebra in \mathcal{C} and the category of its left modules in \mathcal{C} is equivalent to \mathcal{M} .

An important fact in the theory of module categories is that the category $\mathcal{C}_{\mathcal{M}}^* := \text{Func}(\mathcal{M}, \mathcal{M})$ of \mathcal{C} -module endofunctors of an indecomposable \mathcal{C} -module category \mathcal{M} (called the *dual* of \mathcal{C} with respect to \mathcal{M}) is also a fusion category and

$$\text{FPdim}(\mathcal{C}_{\mathcal{M}}^*) = \text{FPdim}(\mathcal{C}).$$

Furthermore, \mathcal{M} is an indecomposable left $\mathcal{C}_{\mathcal{M}}^*$ -module category. If \mathcal{M} is the category of left modules over an algebra A in \mathcal{C} then $\mathcal{C}_{\mathcal{M}}^*$ is equivalent to the fusion category of A -bimodules in \mathcal{C} with the tensor product \otimes_A , see [O1].

Following M. Müger [Mu] we will say that two fusion categories \mathcal{C} and \mathcal{D} are *Morita equivalent* if there is a right \mathcal{C} -module category \mathcal{M} such that $\mathcal{D} \cong \mathcal{C}_{\mathcal{M}}^*$. It was shown in [Mu] that the above relation is indeed an equivalence. For a fixed \mathcal{M} the assignment

$$(2) \quad \mathcal{N} \mapsto \text{Func}(\mathcal{M}, \mathcal{N})$$

establishes an equivalence between 2-categories of right \mathcal{C} -module categories and right $\mathcal{C}_{\mathcal{M}}^*$ -module categories.

Define a *rank* of a semisimple category \mathcal{A} to be the number of equivalence classes of simple objects in \mathcal{A} . A fusion category of rank 1 is equivalent to the category Vec of k -vector spaces. A module category of rank 1 over a fusion category \mathcal{C} is the same thing as a *fiber functor*, i.e., a tensor functor $F : \mathcal{C} \rightarrow \text{Vec}$. From such a functor F one obtains a semisimple Hopf algebra $H := \text{End}(F)$ such that \mathcal{C} is equivalent to the category $\text{Rep}(H)$ of finite-dimensional representations of H [U].

A fusion category is called *pointed* if every its simple object is invertible. Every pointed fusion category is equivalent to a category Vec_G^ω , where G is a finite group and $\omega \in Z^3(G, k^\times)$ is a 3-cocycle. By definition, the latter is the category of G -graded vector spaces with the associativity constraint given by ω . The simple objects of Vec_G^ω are 1-dimensional G -graded vector spaces which will be denoted by g , $g \in G$. Note that the rank and Frobenius-Perron dimension of Vec_G^ω are equal to $|G|$.

Definition 2.1. Let \mathcal{C} be a fusion category and let \mathcal{M} be an indecomposable \mathcal{C} -module category. We will say that \mathcal{M} is *pointed* if $\mathcal{C}_\mathcal{M}^*$ is pointed.

Lemma 2.2. Let \mathcal{C} be a fusion category and let $\mathcal{D} \subseteq \mathcal{C}$ be a fusion subcategory. Let \mathcal{M} be a pointed \mathcal{C} -module category and let $\mathcal{M} = \bigoplus_{x \in S} \mathcal{M}_x$ be its decomposition into a direct sum of indecomposable \mathcal{D} -module categories. Then $\mathcal{M}_x \cong \mathcal{M}_y$ as \mathcal{D} -module categories for all $x, y \in S$.

Proof. Any \mathcal{C} -module autoequivalence of \mathcal{M} induces a permutation of S . Since \mathcal{M} is indecomposable as a module category over the pointed category $\mathcal{C}_\mathcal{M}^*$, the simple objects of $\mathcal{C}_\mathcal{M}^*$ act transitively on S . Hence, for any pair $x, y \in S$ there is a \mathcal{C} -module autoequivalence $F : \mathcal{M} \rightarrow \mathcal{M}$ that maps \mathcal{M}_x to \mathcal{M}_y . \square

2.2. Group-theoretical fusion categories. A fusion category that has a pointed module category is called *group-theoretical* [O2, ENO]. Any such category is equivalent to the category of bimodules over an algebra in Vec_G^ω for some G and $\omega \in Z^3(G, k^\times)$. Below we recall a description of group-theoretical categories from [O2].

Let G be a finite group and $\omega \in Z^3(G, k^\times)$. Equivalence classes of indecomposable right Vec_G^ω -module categories correspond to pairs (H, μ) , where H is a subgroup of G such that $\omega|_{H \times H \times H}$ is cohomologically trivial and $\mu \in C^2(H, k^\times)$ is a 2-cochain satisfying $\delta^2 \mu = \omega|_{H \times H \times H}$. The corresponding Vec_G^ω -module category is constructed as follows. Given a pair (H, μ) as above define an algebra

$$R(H, \mu) = \bigoplus_{a \in H} a$$

in Vec_G^ω with the multiplication

$$(3) \quad \bigoplus_{a, b \in H} \mu(a, b) \text{id}_{ab} : R(H, \mu) \otimes R(H, \mu) \rightarrow R(H, \mu).$$

Let $\mathcal{M}(H, \mu)$ denote the category of left $R(H, \mu)$ -modules in Vec_G^ω . Any Vec_G^ω -module category is equivalent to some $\mathcal{M}(H, \mu)$. The rank of $\mathcal{M}(H, \mu)$ is equal to the index of H in G . Two Vec_G^ω -module categories $\mathcal{M}(H, \mu)$ and $\mathcal{M}(H', \mu')$ are equivalent if and only if there is $g \in G$ such that $H' = gHg^{-1}$ and μ and the g -conjugate of μ' differ by a coboundary.

Let us analyze $(\text{Vec}_G^\omega)^*_{\mathcal{M}(H, \mu)}$ -module categories using correspondence (2). Let (H_1, μ_1) and (H_2, μ_2) be two pairs as above. The rank of the semisimple category $\text{Fun}_{\text{Vec}_G^\omega}(\mathcal{M}(H_1, \mu_1), \mathcal{M}(H_2, \mu_2))$ was computed in [O2, Proposition 3.1]. Namely, for any $g \in G$ the group $H^g := H_1 \cap gH_2g^{-1}$ has a well-defined 2-cocycle

$$\begin{aligned} \mu^g(h, h') &:= \mu_1(h, h') \mu_2(g^{-1}h'^{-1}g, g^{-1}h^{-1}g) \omega(hh'g, g^{-1}h'^{-1}g, g^{-1}h^{-1}g)^{-1} \times \\ &\quad \omega(h, h', g) \omega(h, h'g, g^{-1}h'^{-1}g), \quad h, h' \in H^g. \end{aligned}$$

The simple objects of $\text{Fun}_{\text{Vec}_G^\omega}(\mathcal{M}(H_1, \mu_1), \mathcal{M}(H_2, \mu_2))$ correspond to pairs (Z, π) , where Z is a two-sided (H_1, H_2) -coset in G and π is an irreducible projective representation of H^g with the Schur multiplier μ^g , $g \in Z$. In particular,

$$(4) \quad \text{rank}(\text{Fun}_{\text{Vec}_G^\omega}(\mathcal{M}(H_1, \mu_1), \mathcal{M}(H_2, \mu_2))) = \sum_{i \in H_1 \backslash G / H_2} m(g_i),$$

where $\{g_i\}_{i \in H_1 \backslash G / H_2}$ is a set of representatives of two-sided (H_1, H_2) -cosets in G and $m(g_i)$ is the number of non-equivalent irreducible projective representations of the group H^{g_i} with the Schur multiplier μ^{g_i} . Note that $m(g_i)$ is independent from the choice of a coset representative g_i .

Remark 2.3. Let us note several consequences of the rank formula (4).

- (i) The category $(\text{Vec}_G^\omega)^*_{\mathcal{M}(H, \mu)}$ is equivalent to the representation category of a Hopf algebra if and only if there is pair (K, ν) , where K is a subgroup of G and $\mu \in H^2(K, k^\times)$ such that $\omega|_{K \times K \times K}$ is trivial, $HK = G$, and $\mu\nu^{-1}|_{H \cap K}$ is non-degenerate. The latter condition means that the group algebra of $H \cap K$ twisted by a cocycle representing $\mu\nu^{-1}|_{H \cap K}$ is simple.

Moreover, the conjugacy classes of pairs (K, ν) with the above properties parameterize fiber functors of the category $(\text{Vec}_G^\omega)^*_{\mathcal{M}(H, \mu)}$.

- (ii) Simple objects of $(\text{Vec}_G^\omega)^*_{\mathcal{M}(H, \mu)}$ correspond to pairs (Z, π) , where Z is a two-sided coset of H in G and π is an irreducible projective representation of $H \cap H^g$ with the Schur multiplier μ^g , $g \in Z$. The Frobenius-Perron dimension of the simple object corresponding to (Z, π) is $\deg(\pi)(|Z|/|H|)$.
- (iii) The pointed $(\text{Vec}_G^\omega)^*_{\mathcal{M}(H, \mu)}$ -module categories are

$$\text{Fun}_{\text{Vec}_G^\omega}(\mathcal{M}(H, \mu), \mathcal{M}(N, \nu)),$$

where N is a normal Abelian subgroup of G such that $\omega|_{N \times N \times N}$ is trivial and $\nu \in H^2(N, k^\times)$ is a G -invariant cohomology class, see [Na].

2.3. Group actions on fusion categories and equivariantization. Let us recall the following well known construction (see [AG, G]): given a fusion category \mathcal{C} equipped with an action of a finite group G one defines a new fusion category, namely *the category of G -equivariant objects* of \mathcal{C} , also known as the *equivariantization* of \mathcal{C} .

Let \mathcal{C} be a fusion category. Consider the category $\underline{\text{Aut}}_\otimes(\mathcal{C})$, whose objects are tensor auto-equivalences of \mathcal{C} and whose morphisms are isomorphisms of tensor functors. The category $\underline{\text{Aut}}_\otimes(\mathcal{C})$ has an obvious structure of monoidal category, in which the tensor product is the composition of tensor functors.

For a finite group G let $\text{Cat}(G)$ denote the monoidal category whose objects are elements of G , the only morphisms are the identities, and the tensor product is given by multiplication in G .

Definition 2.4. An *action* of a group G on a fusion category \mathcal{C} is a monoidal functor

$$\text{Cat}(G) \rightarrow \underline{\text{Aut}}_\otimes(\mathcal{C}) : g \mapsto T_g.$$

In this situation we also say that G *acts* on \mathcal{C} .

Let G be a finite group acting on a fusion category \mathcal{C} . For any $g \in G$ let $T_g \in \underline{\text{Aut}}_\otimes(\mathcal{C})$ be the corresponding functor and for any $g, h \in G$ let $\gamma_{g,h}$ be the isomorphism $T_g \circ T_h \simeq T_{gh}$ that defines the monoidal structure on the functor $\text{Cat}(G) \rightarrow \underline{\text{Aut}}_\otimes(\mathcal{C})$.

Definition 2.5. A G -equivariant object in \mathcal{C} is an object X of \mathcal{C} together with isomorphisms $u_g : T_g(X) \simeq X, g \in G$, such that the diagram

$$\begin{array}{ccc} T_g(T_h(X)) & \xrightarrow{T_g(u_h)} & T_g(X) \\ \gamma_{g,h}(X) \downarrow & & \downarrow u_g \\ T_{gh}(X) & \xrightarrow{u_{gh}} & X \end{array}$$

commutes for all $g, h \in G$. One defines morphisms of equivariant objects to be morphisms in \mathcal{C} commuting with $u_g, g \in G$.

The category of G -equivariant objects of \mathcal{C} , or *equivariantization*, will be denoted by \mathcal{C}^G . It has an obvious structure of a fusion category and a forgetful tensor functor $\text{Forg} : \mathcal{C}^G \rightarrow \mathcal{C}$.

Remark 2.6. If \mathcal{C} has a fiber functor $F : \mathcal{C} \rightarrow \text{Vec}$ then $F \circ \text{Forg}$ is a fiber functor of \mathcal{C}^G . Thus, if \mathcal{C} is a representation category of a Hopf algebra then so is \mathcal{C}^G .

Example 2.7. Let G, K be finite groups such that G acts on K by automorphisms. Then G acts on the category $\text{Rep}(K)$ of representations of K and $\text{Rep}(K)^G \cong \text{Rep}(K \rtimes G)$.

2.4. Tambara-Yamagami categories. In [TY] D. Tambara and S. Yamagami completely classified all $\mathbb{Z}/2\mathbb{Z}$ -graded fusion categories in which all but one of simple objects are invertible. They showed that any such category $\mathcal{TY}(A, \chi, \tau)$ is determined, up to an equivalence, by a finite Abelian group A , an isomorphism class of a non-degenerate symmetric bilinear form $\chi : A \times A \rightarrow k^\times$, and a number $\tau \in k$ such that $\tau^2 = |A|^{-1}$. The category $\mathcal{TY}(A, \chi, \tau)$ is described as follows. It is a skeletal category (i.e., such that isomorphic objects are equal) with simple objects $a, a \in A$, and m and tensor product

$$a \otimes b = a + b, \quad a \otimes m = m, \quad m \otimes a = m, \quad m \otimes m = \oplus_{a \in A} a,$$

for all $a, b \in A$ and the unit object $0 \in A$. The associativity constraints

$$\alpha_{x,y,z} : (x \otimes y) \otimes z \cong x \otimes (y \otimes z),$$

where x, y, z are objects of $\mathcal{TY}(A, \chi, \tau)$, are given by

$$\begin{aligned} \alpha_{a,b,c} &= \text{id}_{a+b+c}, \\ \alpha_{a,b,m} &= \text{id}_m, \\ \alpha_{a,m,b} &= \chi(a, b) \text{id}_m, \\ \alpha_{m,a,b} &= \text{id}_m, \\ \alpha_{a,m,m} &= \oplus_{b \in A} \text{id}_b, \\ \alpha_{m,a,m} &= \oplus_{b \in A} \chi(a, b) \text{id}_b, \\ \alpha_{m,m,a} &= \oplus_{b \in A} \text{id}_b, \\ \alpha_{m,m,m} &= \oplus_{a,b \in A} \tau \chi(a, b)^{-1} \text{id}_m. \end{aligned}$$

The unit constraints are the identity maps. The category $\mathcal{TY}(A, \chi, \tau)$ is rigid with $a^* = -a$ and $m^* = m$.

The Frobenius-Perron dimensions of simple objects of $\mathcal{TY}(A, \chi, \tau)$ are $\text{FPdim}(a) = 1, a \in A$, and $\text{FPdim}(m) = \sqrt{|A|}$. We have $\text{FPdim}(\mathcal{TY}(A, \chi, \tau)) = 2|A|$.

Definition 2.8. Let A be an Abelian group and let $\chi : A \times A \rightarrow k^\times$ be a non-degenerate symmetric bilinear form on it. The form χ is called *hyperbolic* if there are subgroups L, L' of A such that $A = L \times L'$ and

$$\chi|_{L \times L} = \chi|_{L' \times L'} = 1.$$

Any subgroup L of A for which there is L' with the above properties is called *Lagrangian* (with respect to χ).

Remark 2.9. Suppose that $|A|$ is odd. It was shown by D. Tambara in [T] that $\mathcal{TY}(A, \chi, \tau)$ admits a fiber functor (i.e., $\mathcal{TY}(A, \chi, \tau)$ is equivalent to the representation category of a semisimple Hopf algebra) if and only if τ^{-1} is a positive integer and χ is hyperbolic.

In the special case when $A = \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z}$, $k \geq 2$, the corresponding semisimple Hopf algebras were explicitly described by G. Kac and V. Paljutkin [KP].

Proposition 2.10. Let $\text{Aut}(A, \chi)$ be the group of automorphisms of A preserving the form χ . There is an action $g \mapsto T_g$ of $\text{Aut}(A, \chi)$ on $\mathcal{TY}(A, \chi, \tau)$, where

$$T_g(A) = g(A), \quad T_g(m) = m, \quad a \in A, g \in \text{Aut}(A, \chi),$$

with the tensor structure of T_g given by identity morphisms.

Proof. This follows directly from the definition of the associativity constraints in the Tambara-Yamagami category and an observation that $\alpha_{T_g(x), T_g(y), T_g(z)} = T_g(\alpha_{x,y,z})$ for all simple objects x, y, z in $\mathcal{TY}(A, \chi, \tau)$ and all $g \in \text{Aut}(A, \chi)$. \square

3. CROSSED PRODUCT FUSION CATEGORIES AND THEIR MODULE CATEGORIES

In this Section we construct a fusion category $\mathcal{C} \rtimes G$ dual to the equivariantization category \mathcal{C}^G (see Proposition 3.2 below) with respect to the module category \mathcal{C} and derive a criterion for \mathcal{C}^G and $\mathcal{C} \rtimes G$ to be group-theoretical.

For a pair of Abelian categories $\mathcal{A}_1, \mathcal{A}_2$, let $\mathcal{A}_1 \boxtimes \mathcal{A}_2$ denote their Deligne's tensor product [D].

3.1. Definition of a crossed product fusion category. Let \mathcal{C} be a fusion category. Fix a finite group G and an action $\text{Cat}(G) \rightarrow \underline{\text{Aut}}_\otimes(\mathcal{C}) : g \mapsto T_g$.

Definition 3.1. A *crossed product* category $\mathcal{C} \rtimes G$ is defined as follows. We set $\mathcal{C} \rtimes G = \mathcal{C} \boxtimes \text{Vec}_G$ as an Abelian category, and define a tensor product by

$$(5) \quad (X \boxtimes g) \otimes (Y \boxtimes h) := (X \otimes T_g(Y)) \boxtimes gh, \quad X, Y \in \mathcal{C}, \quad g, h \in G.$$

The unit object is $\mathbf{1} \boxtimes e$ and the associativity and unit constraints come from those of \mathcal{C} .

Note that $\mathcal{C} \rtimes G$ is a G -graded fusion category,

$$\mathcal{C} \rtimes G = \bigoplus_{g \in G} (\mathcal{C} \rtimes G)_g, \quad \text{where } (\mathcal{C} \rtimes G)_g = \mathcal{C} \otimes (\mathbf{1} \boxtimes g).$$

In particular, $\mathcal{C} \rtimes G$ contains $\mathcal{C} = \mathcal{C} \otimes (\mathbf{1} \boxtimes e)$ as a fusion subcategory.

We have $\text{FPdim}(\mathcal{C} \rtimes G) = |G| \text{FPdim}(\mathcal{C})$.

There is a left $(\mathcal{C} \rtimes G)$ -module category structure on \mathcal{C} given by

$$V \otimes (X \boxtimes g) := T_{g^{-1}}(V \otimes X), \quad V, X \in \mathcal{C}, \quad g \in G,$$

with the associativity constraint

$$\begin{aligned}
\alpha_{V, X \boxtimes g, Y \boxtimes h} : V \otimes ((X \boxtimes g) \otimes (Y \boxtimes h)) &= T_{h^{-1}g^{-1}}(V \otimes (X \otimes T_g(Y))) \\
&\cong T_{h^{-1}}(T_{g^{-1}}(V \otimes X) \otimes Y) \\
&= T_{g^{-1}}(V \otimes X) \otimes (Y \boxtimes h) \\
&= (V \otimes (X \boxtimes g)) \otimes (Y \boxtimes h).
\end{aligned}$$

Proposition 3.2. *We have $(\mathcal{C} \rtimes G)_{\mathcal{C}}^* \cong \mathcal{C}^G$, i.e., the categories $(\mathcal{C} \rtimes G)$ and \mathcal{C}^G are Morita equivalent.*

Proof. Let $F : \mathcal{C} \rightarrow \mathcal{C}$ be a $(\mathcal{C} \rtimes G)$ -module functor. In particular, F is a \mathcal{C} -module functor, hence $F(V) = X \otimes V$ for some X in \mathcal{C} . It is straightforward to check that a $(\mathcal{C} \rtimes G)$ -module functor structure on the latter functor is the same thing as a G -equivariant object structure on X . \square

3.2. A criterion for a crossed product fusion category to be group theoretical. Let \mathcal{C} be a fusion category, let $t \in \underline{\text{Aut}}_{\otimes}(\mathcal{C})$ be a tensor autoequivalence of \mathcal{C} , and let \mathcal{M} be a \mathcal{C} -module category.

Let \mathcal{M}^t denote the module category obtained from \mathcal{M} by twisting the multiplication by means of t , i.e., by defining a new action of \mathcal{C} :

$$M \otimes^t X := M \otimes t(X),$$

for all objects M in \mathcal{M} and X in \mathcal{C} . If A is an algebra in \mathcal{C} such that \mathcal{M} is equivalent to the category of A -modules in \mathcal{C} then \mathcal{M}^t is equivalent to the category of $t(A)$ -modules in \mathcal{C} .

Lemma 3.3. *Let \mathcal{M} be a \mathcal{C} -module category and let $t \in \underline{\text{Aut}}_{\otimes}(\mathcal{C})$ be a tensor autoequivalence of \mathcal{C} . Then $\mathcal{C}_{\mathcal{M}}^* \cong \mathcal{C}_{\mathcal{M}^t}^*$. In particular, if \mathcal{M} is pointed, then so is \mathcal{M}^t .*

Proof. Let $F : \mathcal{M} \rightarrow \mathcal{M}$ be a \mathcal{C} -module functor with the \mathcal{C} -module functor structure given by

$$\gamma_{M,X} : F(M \otimes X) \cong F(M) \otimes X, \quad M \in \mathcal{M}, X \in \mathcal{C}.$$

Let $F^t : \mathcal{M}^t \rightarrow \mathcal{M}^t$ be a \mathcal{C} -module functor defined by $F^t(M) = F(M)$ with a \mathcal{C} -module functor structure

$$\gamma_{M,X}^t : F^t(M \otimes^t X) = F(M \otimes t(X)) \xrightarrow{\gamma_{M,t(X)}} F(M) \otimes t(X) = F^t(M) \otimes^t X.$$

It is straightforward to check that $F \mapsto F^t$ is a tensor equivalence between $\mathcal{C}_{\mathcal{M}}^*$ and $\mathcal{C}_{\mathcal{M}^t}^*$. \square

Given an action $g \mapsto T_g$ of a group G on \mathcal{C} we will denote by \mathcal{M}^g the category \mathcal{M}^{T_g} . We have $\mathcal{M}^{gh} \cong (\mathcal{M}^g)^h$, $g, h \in G$, i.e., G acts on the set of indecomposable \mathcal{C} -module categories.

Definition 3.4. We will say that a \mathcal{C} -module category \mathcal{M} is G -invariant if $\mathcal{M} \cong \mathcal{M}^g$ for every $g \in G$.

Theorem 3.5. *The category $\mathcal{C} \rtimes G$ is group-theoretical if and only if there exists a G -invariant pointed \mathcal{C} -module category.*

Proof. Let \mathcal{M} be a $(\mathcal{C} \rtimes G)$ -module category and let $\mathcal{M} = \oplus_{x \in S} \mathcal{M}_x$ be a decomposition of \mathcal{M} into a direct sum of indecomposable \mathcal{C} -module categories. Note that $\mathcal{M}_{gx} := \mathcal{M}_x \otimes (\mathbf{1} \boxtimes g)$, $x \in S$, $g \in G$, is a \mathcal{C} -module subcategory of \mathcal{M} . This makes

S a transitive G -set. The functor $M \mapsto M \otimes (1 \boxtimes g)$ is a \mathcal{C} -module equivalence between $(\mathcal{M}_x)^g$ and \mathcal{M}_{gx} .

Suppose that \mathcal{M} is pointed, then $\mathcal{M}_x \cong \mathcal{M}_y$ for all $x, y \in S$ by Lemma 2.2. Thus, for any $x \in S$ the category \mathcal{M}_x is G -invariant.

By [ENO, Proposition 5.3] there is a surjective tensor functor

$$(\mathcal{C} \rtimes G)_{\mathcal{M}}^* \rightarrow \mathcal{C}_{\mathcal{M}}^* = \bigoplus_{x, y \in S} \text{Func}(\mathcal{M}_x, \mathcal{M}_y).$$

This means, in particular, that every simple object in $\mathcal{C}_{\mathcal{M}_x}^*$ ($x \in S$) is a subobject of the image of an invertible object in $(\mathcal{C} \rtimes G)_{\mathcal{M}}^*$ and, hence, is invertible. Thus, \mathcal{M}_x is pointed.

To prove the converse implication, suppose that \mathcal{N} is a G -invariant pointed \mathcal{C} -module category. Choose right \mathcal{C} -module equivalences $K_g : \mathcal{N}^g \cong \mathcal{N}$, $g \in G$. We have natural isomorphisms

$$K_g(N \otimes X) \cong K_g(N) \otimes T_g^{-1}(X), \quad N \in \mathcal{N}, X \in \mathcal{C}.$$

We equip $\mathcal{M} := \mathcal{N} \boxtimes \text{Vec}_G$ with a $(\mathcal{C} \rtimes G)$ -module category structure by setting

$$(N \boxtimes f) \otimes (X \boxtimes g) := K_{fg} K_f^{-1}(N \otimes X) \boxtimes fg, \quad N \in \mathcal{N}, X \in \mathcal{C}, f, g \in G.$$

The associativity constraint of \mathcal{M} is defined via the following isomorphisms:

$$\begin{aligned} (N \boxtimes f) \otimes ((X \boxtimes g) \otimes (Y \boxtimes h)) &= K_{fgh} K_f^{-1}(N \otimes (X \otimes T_g(Y))) \boxtimes fgh \\ &\cong K_{fgh} K_f^{-1}((N \otimes X) \otimes T_g(Y)) \boxtimes fgh \\ &\cong K_{fgh} K_{fg}^{-1}(K_{fg} K_f^{-1}(N \otimes X) \otimes Y) \boxtimes fgh \\ &= ((N \boxtimes f) \otimes (X \boxtimes g)) \otimes (Y \boxtimes h). \end{aligned}$$

Let us consider the category $(\mathcal{C} \rtimes G)_{\mathcal{M}}^*$. For any right \mathcal{C} -module equivalence $F : \mathcal{N} \cong \mathcal{N}^h$, $h \in G$, define a functor $F_h : \mathcal{M} \rightarrow \mathcal{M}$ by

$$F_h(N \boxtimes f) = K_{hf} F K_f^{-1}(N) \boxtimes hf, \quad N \in \mathcal{N}, f, h \in G.$$

Then F_h has a structure of a right $(\mathcal{C} \rtimes G)$ -module endofunctor of \mathcal{M} :

$$\begin{aligned} F_h((N \boxtimes f) \otimes (X \boxtimes g)) &= K_{hfg} F K_f^{-1}(N \otimes X) \boxtimes hfg \\ &\cong K_{hfg} F(K_f^{-1}(N) \otimes T_f(X)) \boxtimes hfg \\ &\cong K_{hfg} K_{hf}^{-1} K_{hf}(F(K_f^{-1}(N)) \otimes T_{hf}(X)) \boxtimes hfg \\ &\cong K_{hfg} K_{hf}^{-1}(K_{hf} F K_f^{-1}(N) \otimes X) \boxtimes hfg \\ &= ((K_{hf} F K_f^{-1}(N) \otimes X) \boxtimes hf) \otimes (X \boxtimes g) \\ &= F_h(N \boxtimes f) \otimes (X \boxtimes g). \end{aligned}$$

Each functor F_h is an equivalence. Since there are $|G| \text{FPdim}(\mathcal{C})$ such functors, we conclude that $(\mathcal{C} \rtimes G)_{\mathcal{M}}^*$ is spanned by invertible objects, i.e., it is pointed. \square

Corollary 3.6. *The category \mathcal{C}^G is group-theoretical if and only if there exists a G -invariant pointed \mathcal{C} -module category.*

4. CONSTRUCTION OF A SERIES OF NON GROUP-THEORETICAL FUSION CATEGORIES FROM TAMBARA-YAMAGAMI CATEGORIES

Let p be an odd prime.

4.1. A group-theoretical category \mathcal{C}_p . Let $G := D_{2p} \times \mathbb{Z}/p\mathbb{Z}$, where D_{2p} is the dihedral group of order $2p$. Let K be a non-normal subgroup of G of order p . Let

$$\mathcal{C}_p := (\text{Vec}_G)^*_{\mathcal{M}(K,1)}.$$

Note that the centralizer of K is the unique Sylow p -subgroup of G . It follows from Section 2.2 (see [O2]) that the category \mathcal{C}_p has p^2 invertible objects and a unique simple object of Frobenius-Perron dimension p . Thus, \mathcal{C}_p is a Tambara-Yamagami fusion category.

Note that \mathcal{C}_p admits a fiber functor, since $G = D_{2p}K$ and $D_{2p} \cap K = \{e\}$, see Remark 2.3(i). It follows from Remark 2.9 (see [T]) that

$$\mathcal{C}_p \cong \mathcal{TY}(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}, \chi, \frac{1}{p}),$$

where χ is a non-degenerate hyperbolic bilinear form on $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Note that such χ is unique up to an automorphism of $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ and so \mathcal{C}_p does not depend on the choice of K .

4.2. Pointed \mathcal{C}_p -module categories.

Proposition 4.1. *\mathcal{C}_p has exactly 4 non-equivalent pointed module categories. Two of these categories have rank $2p$ and two others have rank 2.*

Proof. Recall from Remark 2.3(ii) (see [Na]) that pointed Vec_G -module categories correspond to pairs (H, ν) , where H is a normal Abelian subgroup of G and $\nu \in H^2(H, k^\times)$ is a G -invariant cohomology class. The normal Abelian subgroups of G are the following: $\{e\}$, two normal subgroups of order p (denoted H_1, H_2), and the Sylow p -subgroup P . The subgroups $\{e\}, H_1, H_2$ have trivial second cohomology, and the only G -invariant cohomology class in P is the trivial one. Hence,

$$\mathcal{M}(\{e\}, 1), \mathcal{M}(H_1, 1), \mathcal{M}(H_2, 1), \mathcal{M}(P, 1)$$

are all the pointed Vec_G -module categories.

The corresponding \mathcal{C}_p -module categories and their ranks are found using Remark 2.3:

$$\begin{aligned} \text{rank}(\text{Fun}_{\text{Vec}_G}(\mathcal{M}(\{e\}, 1), \mathcal{M}(K, 1))) &= 2p, \\ \text{rank}(\text{Fun}_{\text{Vec}_G}(\mathcal{M}(P, 1), \mathcal{M}(K, 1))) &= 2p, \\ \text{rank}(\text{Fun}_{\text{Vec}_G}(\mathcal{M}(H_1, 1), \mathcal{M}(K, 1))) &= 2, \\ \text{rank}(\text{Fun}_{\text{Vec}_G}(\mathcal{M}(H_2, 1), \mathcal{M}(K, 1))) &= 2, \end{aligned}$$

and the statement follows. \square

Fix a primitive p -th root of unity ξ in k . Any hyperbolic form χ on $A := \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} = \{(x, y) \mid x, y \in \mathbb{Z}/p\mathbb{Z}\}$ is isomorphic to

$$(6) \quad \chi((x_1, x_2), (y_1, y_2)) = \xi^{x_1 y_2 + y_1 x_2},$$

see, e.g., [S]. There are exactly two Lagrangian subgroups of A :

$$(7) \quad L_1 = \{(x, 0) \mid x \in \mathbb{Z}/p\mathbb{Z}\} \quad \text{and} \quad L_2 = \{(0, y) \mid y \in \mathbb{Z}/p\mathbb{Z}\}.$$

Observe that there is a $\mathbb{Z}/2\mathbb{Z}$ -grading

$$\mathcal{C}_p = (\mathcal{C}_p)_0 \oplus (\mathcal{C}_p)_1,$$

where the invertible objects of \mathcal{C}_p span the trivial component $(\mathcal{C}_p)_0 \cong \text{Vec}_A$ and the unique non-invertible simple object spans $(\mathcal{C}_p)_1 \cong \text{Vec}$.

The next two Lemmas describe the pointed \mathcal{C}_p -module categories from Proposition 4.1 in terms of the explicit presentation of Tambara-Yamagami categories given in Section 2.4.

Recall that for a subgroup $H \subset A$ and a 2-cocycle $\mu \in Z^2(H, k^\times)$ we defined the algebra $R(H, \mu) = \bigoplus_{a \in H} a$ in equation (3) in Section 2.2.

Lemma 4.2. *Consider algebras $R_i := R(L_i, 1)$, $i = 1, 2$ in $\text{Vec}_A \subset \mathcal{C}_p$. The categories of R_i -modules, $i = 1, 2$, in \mathcal{C}_p are non-equivalent pointed \mathcal{C}_p -module categories of rank $2p$.*

Proof. Fix $i \in \{1, 2\}$. There are p non-isomorphic simple R_i -modules and p^2 non-isomorphic simple R_i -bimodules in $(\mathcal{C}_p)_0$. The object m has p structures of an R_i -module and, hence, p^2 non-isomorphic structures of an R_i -bimodule, thanks to the associativity constraint property $\alpha_{a,m,b} = \text{id}_m$ for all $a, b \in L_i$. Thus, the category of R_i -modules in \mathcal{C}_p is a pointed \mathcal{C}_p -module category of rank $2p$. (see [O2])

The two \mathcal{C}_p -module categories in question are not equivalent, since by Lemma 2.2 they are already not equivalent as Vec_A -module categories. \square

Lemma 4.3. *Let \mathcal{M} be an indecomposable \mathcal{C}_p -module category of rank 2. There is a cohomologically non-trivial 2-cocycle $\mu \in Z^2(A, k^\times)$ such that \mathcal{M} is equivalent to the category of $R(A, \mu)$ -modules in \mathcal{C}_p .*

Proof. Let x, y be simple objects of \mathcal{M} . It is easy to see that the fusion rules of \mathcal{M} are

$$x \otimes a = x, y \otimes a = y, x \otimes m = py, y \otimes m = px, \quad a \in A.$$

Any such category \mathcal{M} is equivalent to the category of B -modules in \mathcal{C}_p , where $B = \underline{\text{Hom}}(x, x)$ is the internal Hom, see Section 2.1 and [O1]. Thus, $B = \bigoplus_{a \in A} a$ as an object of \mathcal{C}_p . Hence, $B = R(A, \mu)$ for some $\mu \in Z^2(A, k^\times)$, where the algebra $R(A, \mu)$ is defined in (3).

Note that μ must be cohomologically non-trivial, since the object m has p^2 structures of an $R(A, 1)$ -module, and so the category of $R(A, 1)$ -modules in \mathcal{C}_p has rank > 2 . \square

4.3. An action of $\mathbb{Z}/2\mathbb{Z}$ on \mathcal{C}_p without invariant pointed \mathcal{C}_p -module categories. As before, let $A = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ and let $\chi : A \times A \rightarrow k^\times$ be a hyperbolic bilinear form defined in (6). Let t be the group automorphism of A defined by

$$(8) \quad t(x, y) = (y, x), \quad x, y \in \mathbb{Z}/p\mathbb{Z}.$$

Then $t \in \text{Aut}(A, \chi)$, i.e., $\chi \circ (t \times t) = \chi$. By Proposition 2.10 this t gives rise to a tensor autoequivalence of \mathcal{C}_p of order 2 (which we will also denote t) and, hence, to a $\mathbb{Z}/2\mathbb{Z}$ -action on \mathcal{C}_p .

Proposition 4.4. *The above action of $\mathbb{Z}/2\mathbb{Z}$ has no invariant pointed \mathcal{C}_p -module categories.*

Proof. By Proposition 4.1, there are exactly 4 non-equivalent pointed \mathcal{C}_p -module categories. These categories are described in Lemmas 4.2 and 4.3 as categories of modules over certain algebras in \mathcal{C}_p .

Note that t permutes Lagrangian subgroups of A , i.e., maps L_1 to L_2 and vice versa. Hence, t maps $R(L_1, 1)$ to $R(L_2, 1)$ and vice versa. It follows from Lemma 4.2 that t permutes two pointed \mathcal{C}_p -module categories of rank $2p$.

Note that t acts on $H^2(A, k^\times) \cong \mathbb{Z}/p\mathbb{Z}$ by taking inverses, i.e., it maps the cohomology class represented by a 2-cocycle μ to that of μ^{-1} . In particular, the algebra $t(R(A, \mu))$ is isomorphic to $R(A, \mu^{-1})$. We claim that t permutes the two pointed \mathcal{C}_p -module categories of rank 2. Indeed, let \mathcal{M} be such a category. By Lemma 4.3, \mathcal{M} is equivalent to the category of $R(A, \mu)$ -modules in \mathcal{C}_p for some cohomologically non-trivial $\mu \in Z^2(A, k^\times)$. By Lemma 3.3 \mathcal{M}^t is pointed. It is equivalent to the category of $R(A, \mu^{-1})$ -modules. Considering \mathcal{M} and \mathcal{M}^t as Vec_A -module categories we have, using Lemma 2.2:

$$(9) \quad \mathcal{M} \cong \mathcal{M}(A, \mu) \oplus \mathcal{M}(A, \mu) \not\cong \mathcal{M}(A, \mu^{-1}) \oplus \mathcal{M}(A, \mu^{-1}) \cong \mathcal{M}^t,$$

where $\mathcal{M}(A, \mu)$ denotes the Vec_A -module category of $R(A, \mu)$ -modules in Vec_A described in Section 2.1. This means that \mathcal{M} and \mathcal{M}^t are non-equivalent as Vec_A -module categories. Hence, they are not equivalent as \mathcal{C}_p -module categories. \square

Remark 4.5. For any cohomologically non-trivial $\mu \in Z^2(A, k^\times)$ let \mathcal{N}_μ denote the category of $R(A, \mu)$ -modules in \mathcal{C}_p . As a Vec_A -module category \mathcal{N}_μ decomposes as

$$N_\mu = \mathcal{M}(A, \mu) \oplus \mathcal{M}(A, \mu'),$$

where the cohomology class of $\mu' \in Z^2(A, k^\times)$ depends on that of μ as follows. Note that $\text{Alt}(\mu)(x, y) := \mu(y, x)\mu(x, y)^{-1}$, $x, y \in A$ is a non-degenerate alternating bilinear form. There is a unique group automorphism $\iota_\mu \in \text{Aut}(A)$ defined by the property

$$\text{Alt}(\mu)(x, \iota_\mu(a)) = \chi(x, a), \quad \text{for all } x \in A.$$

Then $\mu'(x, y) = \mu^{-1}(\iota_\mu(x), \iota_\mu(y))$, $x, y \in A$. It can be checked directly that there are exactly two cohomology classes μ with the property that μ and μ' are cohomologous and that these two classes are inverses of each other. This fact is not needed in the proof of Proposition 4.4 but rather explains the nature of the cohomology classes of cocycles $\mu, \mu^{-1} \in Z^2(A, k^\times)$ corresponding to pointed \mathcal{C}_p -module categories of rank 2.

Corollary 4.6. *The category $(\mathcal{C}_p)^{\mathbb{Z}/2\mathbb{Z}}$ corresponding to the action (8) is a non group-theoretical fusion category and is equivalent to the representation category of a semisimple Hopf algebra of dimension $4p^2$.*

Proof. The category $(\mathcal{C}_p)^{\mathbb{Z}/2\mathbb{Z}}$ is non group-theoretical by Corollary 3.6 and Proposition 4.4. Since \mathcal{C}_p has a fiber functor, then so does its equivariantization, see Remark 2.6. By Tannakian reconstruction theorem [U], $(\mathcal{C}_p)^{\mathbb{Z}/2\mathbb{Z}}$ is equivalent to the representation category of a semisimple Hopf algebra. \square

5. NON GROUP-THEORETICAL SEMISIMPLE HOPF ALGEBRAS OF DIMENSION $4p^2$ AS EXTENSIONS

Let G be a finite group. Below k^G will denote the commutative Hopf algebra of functions on G and kG will denote the cocommutative group Hopf algebra of G . For a Hopf algebra H its representation category will be denoted $\text{Rep}(H)$.

We will freely use Hopf algebra notation and terminology, see [M] as a reference.

Let H_p be a semisimple Hopf algebra such that $\text{Rep}(H_p) \cong (\mathcal{C}_p)^{\mathbb{Z}/2\mathbb{Z}}$ as a fusion category. We have $\dim_k(H_p) = 4p^2$. In Proposition 5.2 below we find the algebra structure of H_p and its dual H_p^* and describe them in terms of Hopf algebra extensions.

Remark 5.1. It was shown by S. Natale [Nt] that a semisimple Hopf algebra H for which there is a short exact sequence of Hopf algebras

$$k \rightarrow k^{G_1} \rightarrow H \rightarrow k^{G_2} \rightarrow k,$$

where G_1, G_2 are finite groups, then $\text{Rep}(H)$ is group-theoretical. Thus, H_p cannot be obtained as an extension of a cocommutative Hopf algebra by a commutative one.

It was shown by A. Masuoka [Ma] that there is a unique, up to an isomorphism, semisimple Hopf algebra A_p of dimension $2p^2$ with exactly p^2 group-like elements. This Hopf algebra is dual to the Kac-Paljutkin Hopf algebra [KP] and there is a short exact sequence of Hopf algebras

$$k \rightarrow k^{\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}} \rightarrow A_p \rightarrow k(\mathbb{Z}/2\mathbb{Z}) \rightarrow k.$$

As algebras,

$$A_p \cong k^{(2p)} \oplus M_2(k)^{\left(\frac{p(p-1)}{2}\right)} \quad \text{and} \quad A_p^* \cong k^{(p^2)} \oplus M_p(k),$$

where $M_n(k)$ denotes the algebra of n -by- n matrices over k .

Proposition 5.2. *Both H_p and H_p^* are extensions of A_p by $k^{\mathbb{Z}/2\mathbb{Z}}$, i.e., each of them fits into a short exact sequence of Hopf algebras*

$$(10) \quad k \rightarrow k^{\mathbb{Z}/2\mathbb{Z}} \rightarrow H \rightarrow A_p \rightarrow k,$$

where $H = H_p$ or $H = H_p^*$.

$$\text{As algebras, } H_p \cong H_p^* \cong k^{(2p)} \oplus M_2(k)^{\left(\frac{p(p-1)}{2}\right)} \oplus M_p(k)^{(2)}.$$

Proof. The isomorphism type of the algebra H_p is found directly by computing Frobenius-Perron dimensions of simple objects of $\text{Rep}(H_p) \cong (\mathcal{C}_p)^{\mathbb{Z}/2\mathbb{Z}}$.

Since the fusion category $(\mathcal{C}_p)^{\mathbb{Z}/2\mathbb{Z}}$ inherits a $\mathbb{Z}/2\mathbb{Z}$ -grading from \mathcal{C}_p , the Hopf algebra H_p contains a central group-like element of order 2 and, hence, a central Hopf subalgebra $k^{\mathbb{Z}/2\mathbb{Z}}$. The representation category of the quotient Hopf algebra $\overline{H}_p = H/H(k^{\mathbb{Z}/2\mathbb{Z}})^+$ is equivalent to a $\mathbb{Z}/2\mathbb{Z}$ -equivariantization of the fusion subcategory $\text{Vec}_{\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}} \cong \text{Rep}(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z})$ of \mathcal{C}_p . By (8) the latter comes from an order 2 group automorphism t of $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. From Example 2.7 we see that $\text{Rep}(\overline{H}_p) \cong \text{Rep}(kG)$, where $G = (\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}$. It follows from the result of P. Schauenburg [Sch] that there is cocycle twist J on the group Hopf algebra kG such that H_p is isomorphic to the deformation $(kG)^J$ of kG by means of J .

Thus, H_p fits into an extension

$$(11) \quad k \rightarrow k^{\mathbb{Z}/2\mathbb{Z}} \rightarrow H_p \rightarrow (kG)^J \rightarrow k.$$

Note that the Hopf algebra $(kG)^J$ is necessarily non cocommutative, since otherwise H_p would be group-theoretical by the result of S. Natale [Nt], see Remark 5.1. Hence, the cocycle twist J must be non-trivial. It follows from the work of P. Etingof and S. Gelaki [EG] that J comes from a non-degenerate 2-cocycle on the Sylow p -subgroup of G . Therefore, the deformed Hopf algebra $(kG)^J$ has exactly p^2 group-like elements. Hence, $(kG)^J \cong A_p$ by [Ma].

Thus, A_p^* is an index 2 (and, hence, normal) Hopf subalgebra of H_p^* . Therefore, H_p^* has exactly two p -dimensional irreducible representations and the central group-like element of order 2 in A_p^* must also be central in H_p^* . The corresponding quotient Hopf algebra $H_p^*/H_p^*(k^{\mathbb{Z}/2\mathbb{Z}})^+$ is non cocommutative and contains a Hopf subalgebra

isomorphic to $A_p^*/A_p^*(k^{\mathbb{Z}/2\mathbb{Z}})^+ \cong k(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z})$. Again, $H_p^*/H_p^*(k^{\mathbb{Z}/2\mathbb{Z}})^+ \cong A_p$ by [Ma] and so H_p^* fits into the same extension (11) as H_p and has the same algebra structure. \square

Corollary 5.3. *The Hopf algebras H_p are upper and lower semisolvable in the sense of S. Montgomery and S. Witherspoon [MW].*

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